No aids are allowed except writing utensils. The result will be posted on the notice board of the department at the latest on Friday, 31 August at 12:00.

1. Determine the type of the surface
   \[ x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_1x_3 + 6x_2x_3 = -2. \]

2. A linear mapping \( F \) from \( \mathbb{R}^3 \) to \( \mathbb{R}^4 \) is defined by \( F(x) = Ax \) where \( x = (x_1, x_2, x_3) \) and \( A \) is the matrix
   \[
   A = \begin{pmatrix}
   1 & 1 & 0 \\
   -1 & 1 & 4 \\
   0 & 1 & 1 \\
   0 & -1 & 1
   \end{pmatrix}.
   \]
   Determine an orthonormal basis in the image \( \text{im} \, F \) and find the distance from the point \((1, 1, 1, 1)\) to \( \text{im} \, F \).

3. Find the maximum and minimum of the quadratic form
   \[ q(x_1, x_2, x_3) = x_1^2 - 4x_2^2 + x_3^2 + 6x_1x_2 + 4x_1x_3 + 6x_2x_3 \]
   on the unit sphere \( x_1^2 + x_2^2 + x_3^2 = 1 \) and also determine the points where the maximum and minimum are attained.

4. Let \( A \) be the matrix
   \[
   A = \frac{1}{9} \begin{pmatrix}
   a_1 & a_2 & a_3 \\
   4 & 7 & -4 \\
   8 & -4 & 1
   \end{pmatrix}.
   \]
   Determine the first row \( a_1, a_2, a_3 \) so that the matrix \( A \) becomes the matrix of a rotation about an axis with respect to a positively oriented orthonormal basis. Find the axis of rotation, the cosine of the angle of rotation and the direction of rotation.

5. State and prove the Cauchy–Schwarz inequality.

6. Determine the rank of the matrix
   \[
   A = \begin{pmatrix}
   x & 0 & 0 & 4 \\
   0 & x & 0 & 0 \\
   0 & 0 & x & 0 \\
   1 & 0 & 0 & x
   \end{pmatrix}
   \]
   for all real numbers \( x \).