Solutions

1. a) The series

$$\sum_{k=1}^{\infty} \frac{(2 + i)^k}{(1 + 2i)^k}$$

is **divergent** since the absolute value of the term is

$$\frac{|2 + i|^k}{|1 + 2i|^k} = \left(\frac{\sqrt{4 + 1}}{\sqrt{1 + 4}}\right)^k = 1$$

so the terms do not tend to zero.

b) The series

$$\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$$

is **convergent**. With $a_k = \frac{(k!)^2}{(2k)!}$ we have

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)!^2(2k)!}{(2k+2)!(k!)^2} = \frac{(k+1)^2}{(2k+2)(2k+1)} \to \frac{1}{4} \text{ as } k \to \infty.$$ 

The convergence now follows from the ratio test since the limit is less than 1.

e) The series

$$\sum_{k=2}^{\infty} \frac{(-1)^kk}{k^2 - 1}$$

is **convergent**, since it satisfies the conditions of Leibniz’ criterion for convergence:

- The terms are of alternate sign.
- The terms tend to zero as $k$ tends to $\infty$
- The absolute value of the terms is a **decreasing function** of $k$. This follows since if

$$f(x) = \frac{x}{x^2 - 1}$$

the derivative of $f$ is

$$f'(x) = \frac{-1 - x^2}{(x^2 - 1)^2} < 0.$$
2. By Euler’s formulas,
\[
\sin 2x \cos 4x = \left(\frac{e^{2ix} - e^{-2ix}}{2i}\right)\left(\frac{e^{4ix} + e^{-4ix}}{2}\right) = \frac{\sin 6x}{2} - \frac{\sin 2x}{2}.
\]
Hence the solution is
\[
u(x, t) = \frac{1}{2}(e^{-108t} \sin 6x - e^{-12t} \sin 2x).
\]

3. a) The Fourier coefficient is
\[
c_n = \frac{1}{4\pi} \int_{-\pi}^{\pi} (e^x - e^{-x})e^{-inx}dx = \frac{1}{4\pi} \left(\left[\frac{e^{(1-in)x}}{1 - in}\right]_{-\pi}^{\pi} - \left[\frac{e^{(-1-in)x}}{1 - in}\right]_{-\pi}^{\pi}\right)
\]
\[
= \frac{1}{4\pi} \left((-1)^n(e^\pi - e^{-\pi}) - (-1)^n(e^\pi - e^{-\pi})\right)
\]
\[
= \frac{(-1)^n \sinh \pi}{2\pi} \frac{2in}{1 + n^2}
\]
Thus the Fourier series of \(u\) is
\[
u(x) = \frac{i \sinh \pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n ne^{inx}}{1 + n^2}
\]

b) By Parseval’s formula we have
\[
\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\nu(x)|^2dx.
\]
The integral is equal to
\[
\frac{1}{8\pi} \int_{-\pi}^{\pi} e^{2x} - 2 + e^{-2x}dx = \frac{1}{8\pi} (e^{2\pi} - 4\pi - e^{-2\pi}).
\]
It follows that
\[
\frac{\sinh^2 \pi}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{n^2}{(1 + n^2)^2} = \frac{1}{8\pi} (e^{2\pi} - 4\pi - e^{-2\pi}).
\]
The sum is symmetric for positive and for negative \(n\) so
\[
\sum_{k=1}^{\infty} \frac{n^2}{(1 + n^2)^2} = \frac{\pi}{16 \sinh^2 \pi} (e^{2\pi} - 4\pi - e^{-2\pi})
\]

4. Assume that \(u\) is given by a power series with a positive radius of convergence, \(u(x) = \sum_{k=0}^{\infty} a_k x^k\). We can differentiate term by term if \(x\) is within the radius of convergence:
\[
u'(x) = \sum_{k=1}^{\infty} ka_k x^{k-1}, \quad \nu''(x) = \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2}
\]
After insertion in the differential equation we get
\[
\sum_{k=2}^{\infty} k(k-1)a_k x^{k-1} + \sum_{k=1}^{\infty} ka_k x^k + \sum_{k=1}^{\infty} ka_k x^{k-1} + \sum_{k=0}^{\infty} 2a_k x^k = 0.
\]
We replace \( k - 1 \) by \( k \) in the first and the third series:

\[
\sum_{k=1}^{\infty} (k + 1) ka_{k+1} x^k + \sum_{k=0}^{\infty} ka_k x^k + \sum_{k=0}^{\infty} (k + 1) a_{k+1} x^k + \sum_{k=0}^{\infty} 2ak x^k = 0.
\]

If the equality is valid for all \( x \) in a neighborhood of 0 then the coefficient for every power of \( x \) is zero. It follows that

\[
a_1 + 2a_0 = 0, \quad \text{and} \quad (k + 1^2)a_{k+1} + (k + 2)a_k = 0, \quad k \geq 1.
\]

The initial value \( u(0) = 1 \) implies that \( a_0 = 1 \) and then \( a_1 = -2 \). The recursive formula for higher coefficients is

\[
a_{k+1} = \frac{-(k + 2)}{(k + 1)^2} a_k.
\]

A calculation of the first coefficients shows that \( a_2 = 3/2, a_3 = -4/3!, a_4 = 5/4! \). The general formula \( a_k = (k + 1)/k! \) is easily proved by induction so

\[
u(x) = \sum_{k=0}^{\infty} \left( -1 \right)^k \frac{(k + 1)x^k}{k!} = \frac{d}{dx} \left( xe^{-x} \right) = e^{-x} - xe^{-x}.
\]

The series is convergent for all \( x \).

5. The left hand side of the equation is a convolution. By taking Fourier transforms on both sides and using formulas 24, 29 and 30, we find that

\[
\frac{2\hat{u}(\xi)}{1 + \xi^2} = \sqrt{2\pi} e^{-\xi^2/2},
\]

or

\[
\hat{u}(\xi) = \frac{\sqrt{2\pi}}{2} (1 + \xi^2) e^{-\xi^2/2}
\]

By the inversion formula (19), the Fourier transform of \( \hat{u} \) is \( 2\pi u(-x) \), so by taking Fourier transforms on both sides again and using formulas 29 and 27 we get

\[
2\pi u(-x) = \frac{\sqrt{2\pi}}{2} e^{-x^2/2} \left( \sqrt{2\pi} e^{-x^2/2} - \sqrt{2\pi} (e^{-x^2/2})' \right),
\]

from which it follows that

\[
u(x) = e^{-x^2/2} \left( 1 - \frac{x^2}{2} \right).
\]

6. a) For \( x = 0 \) all the terms are zero. For \( x \neq 0 \), we have

\[
\left| \frac{x}{1 + k^2x^2} \right| \leq \frac{1}{k^2x}
\]

so the series is convergent for \( x \neq 0 \) since the sum \( \sum 1/k^2 \) is convergent.

b) Let \( a > 0 \). The supremum of the term \( \frac{x}{1 + k^2x^2} \) for \( x \geq a \) is less than or equal to \( \frac{1}{ak^2} \) and the series

\[
\sum_{k=1}^{\infty} \frac{1}{ak^2}
\]

is convergent since \( a \neq 0 \). By Weierstrass’ M-test, the series is uniformly convergent for \( |x| \geq a \). Therefore, the function \( s \) is continuous for \( x \geq a \). Since this is true for any \( a > 0 \), it follows that \( s \) is continuous for \( x \neq 0 \).

Please, turn over!
c) We have \( s(0) = 0 \). The function \( s \) is continuous at 0 if and only if the limit of \( s(x) \) when \( x \to 0 \) is equal to 0. Let \( N \) be any positive integer. Since the terms of the series are positive we have

\[
\sum_{k=1}^{N} \frac{1}{1 + k^2 / N^2} \geq \frac{1}{N} \sum_{k=1}^{N} \frac{1}{1 + 1} = \frac{1}{2}.
\]

Therefore the limit is not 0 and the function is not continuous at 0.