1. Which of the following series are convergent?
   a) $\sum_{k=1}^{\infty} \frac{\sqrt{k} + 1}{2k^2 - 1}$,  
   b) $\sum_{k=1}^{\infty} \frac{\sqrt{k} + 1}{2k - 1}$,  
   c) $\sum_{k=1}^{\infty} (-1)^k \frac{\sqrt{k} + 1}{2k - 1}$.

2. Find a power series solution of the problem
   
   $$xy'' + (1 - 2x)y' - y = 0, \quad y(0) = 1.$$  

3. Solve the heat conduction problem

   $$\partial_t u(x, t) = 4\partial^2_x u(x, t), \quad 0 \leq x \leq \pi, \quad t > 0,$$
   $$\partial_x u(0, t) = \partial_x u(\pi, t) = 0, \quad t > 0,$$
   $$u(x, 0) = \cos x \cos 3x, \quad 0 \leq x \leq \pi.$$  

4. Show that the sequence $(f_n)_{n=1}^{\infty}$ is uniformly convergent in the interval $[0, \infty)$ where
   
   $$f_n(x) = x^2 e^{-nx}, \quad x \geq 0.$$  

5. Let $a$ be a real number in the open interval $(0, 1)$, and let $u$ be the $2\pi$-periodic function for which $u(x) = \cos ax$ when $-\pi \leq x \leq \pi$.
   a) Find the Fourier series expansion of $u$.
   b) Prove that

   $$\sum_{n=1}^{\infty} \frac{2a}{a^2 - n^2} = \pi \cot a\pi - \frac{1}{a}, \quad 0 < a < 1.$$  
   
   c) Integrate both sides with respect to $a$, and conclude that

   $$\sum_{n=1}^{\infty} \ln \left(1 - \frac{a^2}{n^2}\right) = \ln \left(\frac{\sin a\pi}{a\pi}\right), \quad 0 < a < 1,$$

   providing proper justification.
   d) Deduce the identity

   $$\frac{\sin a\pi}{a\pi} = \prod_{n=1}^{\infty} \left(1 - \frac{a^2}{n^2}\right), \quad 0 < a < 1.$$  

Please, turn over!
and, by means thereof, Wallis’ product formula

\[
\frac{\pi}{2} = \frac{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot \cdots}{1 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdots}.
\]